

Geometry I, 2006 : Coursework 6 Solutions

[Note: Throughout these solutions, $\underline{i}, \underline{j}, \underline{k}$ is a right-handed triple of pairwise orthogonal unit vectors.]

1. [Compare with question 6 on coursework 5.]

[1 mark] Let $P = (1, -2, 3)$, $Q = (-5, 0, 1)$, $R = (2, 1, -1)$, let $\underline{u}$ be represented by $\overrightarrow{PQ}$ and let $\underline{v}$ be represented by $\overrightarrow{PR}$. Then $\underline{u} = \begin{pmatrix} -6 \\ 2 \\ -2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

The required equation is a Cartesian equation of the plane through $P = (1, -2, 3)$ and parallel to $\underline{u}$ and $\underline{v}$.

[1 mark] A vector equation for this plane is $\underline{r} \cdot \underline{n} = \underline{p} \cdot \underline{n}$, where $\underline{n} = \underline{u} \times \underline{v}$ and $\underline{p}$ is the position vector of P .

We have $\underline{n} = \underline{u} \times \underline{v} = -2\underline{i} - 26\underline{j} - 20\underline{k}$, and so $\underline{p} \cdot \underline{n} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -26 \\ -20 \end{pmatrix} = -10$.

Thus, a vector equation for the plane is $\underline{r} \cdot \begin{pmatrix} -2 \\ -26 \\ -20 \end{pmatrix} = -10$, and a Cartesian equation is $-2x - 26y - 20z = -10$, equivalently $x + 13y + 10z = 5$.

2. Let $a = |\underline{u} \times \underline{v}|^2 + (\underline{u} \cdot \underline{v})^2$.

[1 mark] If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$ then $a = 0$.

[2 marks] Otherwise, let θ be the angle between $\underline{u}$ and $\underline{v}$. Then $a = (|\underline{u}||\underline{v}| \sin \theta)^2 + (|\underline{u}||\underline{v}| \cos \theta)^2 = |\underline{u}|^2 |\underline{v}|^2 ((\sin \theta)^2 + (\cos \theta)^2) = |\underline{u}|^2 |\underline{v}|^2$.

We conclude that $|\underline{u} \times \underline{v}|^2 + (\underline{u} \cdot \underline{v})^2 = |\underline{u}|^2 |\underline{v}|^2$ for all vectors $\underline{u}, \underline{v}$.

3. [3 marks; 2 marks for nearly correct proof; 1 mark for something non-trivial correct]

$$\begin{aligned} \underline{u} \times (\underline{v} + \underline{w}) &= -((\underline{v} + \underline{w}) \times \underline{u}) \\ &= -((\underline{v} \times \underline{u}) + (\underline{w} \times \underline{u})) \text{ (by assumed right-distributivity)} \\ &= -(-(\underline{u} \times \underline{v}) - (\underline{u} \times \underline{w})) = (\underline{u} \times \underline{v}) + (\underline{u} \times \underline{w}). \end{aligned}$$

4. (i) [2 marks] This volume is $|\underline{u} \cdot (\underline{v} \times \underline{w})|$. We have $\underline{v} \times \underline{w} = 16\underline{i} - \underline{j} + 7\underline{k} = \begin{pmatrix} 16 \\ -1 \\ 7 \end{pmatrix}$,

and so the volume is $\left| \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 16 \\ -1 \\ 7 \end{pmatrix} \right| = |-16 - 2 + 7| = |-11| = 11$.

(ii) [1 mark] $\underline{u}, \underline{v}, \underline{w}$ are not coplanar, since their triple scalar product is non-zero.

[1 mark] In fact, $\underline{u}, \underline{v}, \underline{w}$ is a left-handed triple, since their triple scalar product is negative.

5. [2 marks each; 1 mark if answer correct, but justification only partly correct; no marks if no justification or justification wrong]

(i) This is false. For example $\underline{i} \neq \underline{0}$, but $\underline{i} \times \underline{i} = \underline{0}$ (indeed, $\underline{u} \times \underline{u} = \underline{0}$ for every vector $\underline{u}$).

(ii) This is true. Suppose $\underline{u} \neq \underline{0}$, and let $\underline{v}$ be a non-zero vector which is not parallel to $\underline{u}$. Such a $\underline{v}$ will always exist, and $\underline{u} \times \underline{v} \neq \underline{0}$. Thus, if $\underline{u} \times \underline{v} = \underline{0}$ for every vector $\underline{v}$ then we must have $\underline{u} = \underline{0}$.

(iii) This is true. If $\underline{u} \times \underline{v} = \underline{v} \times \underline{u}$, then $\underline{u} \times \underline{v} = \underline{v} \times \underline{u} = -(\underline{u} \times \underline{v})$, hence $2(\underline{u} \times \underline{v}) = \underline{0}$, and so $\underline{u} \times \underline{v} = \underline{0}$.

(iv) This is false. For example, $(\underline{i} \times \underline{i}) \times \underline{j} = \underline{0} \times \underline{j} = \underline{0}$, but $\underline{i} \times (\underline{i} \times \underline{j}) = \underline{i} \times \underline{k} = -\underline{j}$.

Leonard Soicher, 17 November 2006